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CONTROLLABILITY AND OBSERVABILITY IN THE PROBLEM OF STABILIZING MECHANICAL SYSTEMS WITH CYCLIC COORDINATES†

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An approach based on linear control theory is used to solve the problem of stabilizing the steady motions of holonomic mechanical systems in which only cyclic coordinates are controllable [1–3]. The most general structure of forces acting on the system is considered and it is assumed that the constraints imposed are time-independent. The set of new criteria of controllability and observability based on the reduction of the problem under consideration is obtained. The reduction enables one to reduce the investigation of these problems to an analysis of a problem of less dimensions.

1. WE CONSIDER a holonomic mechanical system with time-independent constraints. Suppose it is described by generalized coordinates $q_1, \ldots, q_n$ among which there are the coordinates q_j $(j=r+1, \ldots, n;\ r<n)$ not occurring explicitly in the expression for the kinetic energy which is assumed to be explicitly time-independent

$$T=T_0+T_1+T_2$$
$$T_0=T_0(q), \quad T_1=\gamma^T(q)\,q^{\cdot}+\delta^T(q)\,\omega$$
$$T_2=\tfrac{1}{2}q^{\cdot T}A(q)\,q^{\cdot}+q^{\cdot T}C(q)\,\omega+\tfrac{1}{2}\omega^T B(q)\,\omega$$

Here

$$q=\|\,q_1q_2\ldots q_r\,\|^T, \quad q^{\cdot}=\|\,q_1^{\cdot}q_2^{\cdot}\ldots q_r^{\cdot}\,\|^T$$
$$\omega=\|\,q_{r+1}^{\cdot}q_{r+2}^{\cdot}\ldots q_n^{\cdot}\,\|^T$$

are column matrices whose elements are the positional coordinates, and positional and pseudocyclic velocities, $A(r\times r)$ and $B(m\times m)$ are positive definite symmetric matrices, $m=n-r$, $C(r\times m)$ is a rectangular matrix, $\gamma^T(1\times r)$ and $\delta^T(1\times m)$ are row matrices, and $T_0(q)$ is a scalar function. The elements of the matrices A, B, C, γ^T and δ^T depend on the positional coordinates only.

The generalized forces corresponding to the positional coordinates are specified and represented by the sum of potential and non-potential forces

$$Q_i(q,q^{\cdot},\omega)=\partial U/\partial q_i+Q_i^N(q,q^{\cdot},\omega), \quad (U=U(q))$$
$$(i=1,2,\ldots,r)$$

The generalized forces corresponding to the pseudocyclic coordinates are represented by the sum of the specified non-potential forces Q_k^N and controlling forces F_k to be chosen

$$Q_k(q,q^{\cdot},\omega)=Q_k^N(q,q^{\cdot},\omega)+F_k(q,q^{\cdot},\omega) \quad (k=r+1,\ r+2,\ldots,n)$$

Information on the current values of the coordinates $q(t)$ and velocities $q^{\cdot}(t)$ and $\omega(t)$ of the system is supplied by measuring $\Sigma=\Sigma(q,q^{\cdot},\omega)$ of dimensions $l\times 1$.

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We assume that, under certain initial conditions, steady motion

$$q(t)=q_0=\text{const},\quad \omega(t)=\omega_0=\text{const} \tag{1.1}$$

of the system is possible.

The quantities q_0 and ω_0 satisfy the equations

$$-\frac{1}{2}\sum_{k,s=r+1}^{n}\left(\frac{\partial b_{ks}}{\partial q_i}\right)_0\omega_{k0}\omega_{s0}-\sum_{k=r+1}^{n}\left(\frac{\partial \delta_k}{\partial q_i}\right)\omega_{k0}-\left(\frac{\partial T_0}{\partial q_i}\right)_0=$$

$$=\left(\frac{\partial U}{\partial q_i}\right)_0+Q_i^N(q_0,0,\omega) \tag{1.2}$$

$$Q_k^N(q_0,0,\omega_0)+F_k(q_0,0,\omega_0)=0\quad (i=1,2,\dots,r,\ k=r+1,\dots,n) \tag{1.3}$$

$$B=\| b_{ks}\|,\quad \delta=\|\delta_{k+1}\dots\delta_n\|^T$$

From now on, a subscript zero means that the value is calculated at $q=q_0$ and $\omega=\omega_0$.

Equations (1.2) and (1.3) defining a set of possible steady motions of the system in the n-dimensional space of the variables q and ω are more complicated than those considered in [4].

One can separate [4] the class of trivial steady motions when Eq. (1.2) has the solution $q=q_0$ for any values of ω_0. The remaining steady motions are called essential.

In particular, for generalized non-potential forces of the form

$$Q_i^N=Q_i^0(q,q^\cdot)+\sum_{k=r+1}^{n}Q_{ik}^1(q,q^\cdot)\omega_k+\frac{1}{2}\sum_{k,s=r+1}^{n}Q_{is}^2(q,q^\cdot)\omega_k\omega_s$$

the conditions for trivial steady motions to exist have the form

$$\left(\frac{\partial U}{\partial q_i}\right)_0+\left(\frac{\partial T}{\partial q_i}\right)_0-Q_i^0(q_0,0)=0$$

$$\left(\frac{\partial b_{ks}}{\partial q_i}\right)_0+Q_{iks}^2(q_0,0)=0,\quad \left(\frac{\partial \delta_k}{\partial q_i}\right)_0+Q_{ik}^1(q_0,0)=0$$

$$(i=1,2,\dots,r;\ k,s=r+1,\dots,n)$$

Let us introduce the deviations $x=q-q_0$ and $\eta=\omega-\omega_0$ and write the linearized Lagrange equations and the equation of measurements in matrix form

$$A_0x^{\cdot\cdot}+D_0x^\cdot+W_0x+C_0\eta^\cdot-P_0^T\eta=0$$

$$B_0\eta^\cdot+C_0^Tx^{\cdot\cdot}+S_0x^\cdot+R_0x+L_0\eta=F_0u \tag{1.4}$$

$$\sigma=H_1x+H_2x^\cdot+H_3\eta \tag{1.5}$$

Here F_0u is the linear part of the control force F, the matrix F_0 has dimensions of $m\times m$, and $\sigma(l\times 1)$ is the linear part of the measurements vector Σ.

The matrices H_1 and H_2 have dimensions of $l\times r$, while the corresponding dimensions of the other matrices are $l\times m$ for H_3, $r\times r$ for A_0, D_0 and W_0, $m\times m$ for B_0 and L_0, $m\times r$ for P_0, R_0, and S_0, $r\times m$ for C_0. The elements of these matrices are given by the expressions

$$A_0=A(q_0),\quad B_0=B(q_0),\quad C_0=C(q_0),\quad D=\| D_{ij}\|$$

$$D_{ij}=-\left[\frac{\partial Q_i^N}{\partial q_j^\cdot}+\sum_{k=r+1}^{n}\left(\frac{\partial C_{ik}}{\partial q_j}-\frac{\partial C_{jk}}{\partial q_i}\right)\omega_k+\left(\frac{\partial \gamma_i}{\partial q_j}-\frac{\partial \gamma_j}{\partial q_i}\right)\right]\Bigg|_0$$

$$W=\| W_{ij}\|,\quad P_0^T=\| P_{i,k-r}\|$$

$$W_{ij}=\left[-\frac{\partial^2 U}{\partial q_i\partial q_j}-\frac{1}{2}\sum_{k,s=r+1}^{n}\frac{\partial^2 b_{ks}}{\partial q_i\partial q_j}\omega_k\omega_s-\right.$$

$$-\frac{\partial^2 T_0}{\partial q_i \partial q_j}-\frac{\partial Q_i^N}{\partial q_j}-\sum_{k=r+1}^{N}\frac{\partial^2 \delta_k}{\partial q_i \partial q_j}\omega_k]|_0$$

$$L_0=\| L_{k-r,\,s-r}\|,\quad L_{k-r,\,s-r}=\frac{\partial Q_k^N}{\partial \omega_s}|_0$$

$$P_{i,\,k-r}=[\frac{\partial Q_i^N}{\partial \omega_k}+\frac{\partial \delta_k}{\partial q_i}+\sum_{s=r+1}^{n}\frac{\partial b_{ks}}{\partial q_i}\omega_s]|_0$$

$$S_{k-r,\,i}=[\sum_{s=r+1}^{n}\frac{\partial b_{ks}}{\partial q_i}\omega_s+\frac{\partial \delta_k}{\partial q_i}-\frac{\partial Q_k^N}{\partial q_i}]|_0$$

$$S_0=\| S_{k-r,\,i}\|,\quad R_0=\| R_{k-r,\,i}\|,\quad R_{k-r,\,i}=-\frac{\partial Q_k^N}{\partial q_i}|_0$$

$$(i,j=1,2,\ldots,r)\ (k,s=r+1,\ldots,n)$$

If the constraints imposed on the system are time-independent ($T_1\equiv 0$, $T_0\equiv 0$), there are not non-potential forces Q_k^N acting at the cyclic coordinates and the forces $Q_j^N\times Q_{jd}$ acting at the positional coordinates are dissipative, then $S_0=P_0$, $R_0=0$ and $L_0=0$ in Eqs (1.4). This case has been considered in [3].

Note that for gyroscopically uncoupled systems (GUS) ($C_0=0$), when $P_0=0$, the control u does not influence the positional coordinates. The matrix P_0 vanishes, in particularly, when the constraints are time-independent ($\delta=0$), there are no non-potential forces ($Q_i^N=0$), and the steady motions are trivial ($\partial b_{ks}/\partial q_i|_0=0$). But, as follows from the expressions for the elements $P_{i,k-r}$, the matrix P_0 can also vanish for an appropriate choice of the non-potential forces Q_i^N acting at the positional coordinates. In the general case, even in a GUS, for trivial steady motions system (1.4) is not separable into two subsystems one of which is insensitive to control. It provides additional possibilities for stabilizing the trivial steady motions of a GUS.

We will discuss the various statements of the problem on controlling mechanical systems with cyclic coordinates in a neighbourhood of a steady motion. We will restrict ourselves to considering problems in which controls are introduced only with respect to cyclic coordinates. This problem was formulated for the first time by Rumyantsev and Lilov [1, 2]. The most well-known problem is problem 1, which consists of specifying the control which ensures asymptotic stability of a steady motion with respect to positional and cyclic velocities [1, 2]. The other statement of the control problem (problem 2) is also possible in the case when the purpose of controlling the system is to provide asymptotic stability of steady motion with a previously specified decrement speed [3]. In the problem of optimal stabilization it is required to choose the controlling actions in such a way as to minimize a certain functional characterizing certain requirements on the system [5]. In this connection by analogy with the statement of the problem of stability with respect to some of the variables, it is also possible to state the problem of stabilization with respect to some of variables [6]. In fact, the problem of stabilizing the steady motions by forces acting on some of the cyclic coordinates can be reduced to this problem [this corresponds to the condition rank $F_0=m_1<m_0$ for Eqs (1.4)]. The similar problem without non-potential forces has been solved by the methods of the theory of stability in [7].

When solving all the problems mentioned above it is first necessary to answer the question of their solvability in principle. The latter reduces to investigating the controllability and stabilizability of system (1.4). The property of stabilizability is related to problem 1 and the property of controllability is related to problem 2. Furthermore, using the analysis of the observability of systems (1.4) and (1.5) it is necessary to specify the rational structure of the measuring information on the state of the system (i.e. on the quantities x, $x^\bullet$ and η) which is necessary to design the stabilizing control. On the basis of certain measuring information it is then possible to design the stabilization algorithm which realizes the properties required for the closed system, for instance, when solving the problem of optimal stabilization or introducing feedback based on estimation of the state vector [3].

2. We will consider the problem of the controllability (stabilizability) of system (1.4). The standard Kalman criterion of controllability implies the need to analyse the rank of an $(n+r)\times(n+r)m$ matrix. The specific structure of system (1.4) enables one to obtain new effective criteria of controllability by reducing the problem.

In the general case, if $m_1<m$, the vector F_0u can be represented in the form

$$F_0 u = F_0 T_u v = \begin{Vmatrix} E_{m_1} & 0 \\ F_{21} & 0 \end{Vmatrix} \begin{Vmatrix} v_1 \\ v_2 \end{Vmatrix} = F_v v_1 \qquad (2.1)$$

$$\left(F_v = \begin{Vmatrix} E_{m_1} \\ F_{21} \end{Vmatrix}\right)$$

by a linear transformation of the control vector $u = T_u v$ $[T_u(m \times m),\ \det T_u \neq 0]$.

The matrices F_v, F_{21} and v_1 have the dimensions $m \times m_1$, $(m - m_1) \times m_1$ and $m_1 \times 1$, respectively.

Theorem 1. System (1.4) of the $(n+r)$th order is controllable (stabilizable) if and only if the condition

$$\operatorname{rank} \begin{Vmatrix} A_0\lambda^2 + D_0\lambda + W_0 & C_0\lambda - P_0^T \\ (C_{21}^T - F_{21}C_{11}^T)\lambda^2 + (S_{21} - F_{21}S_{11})\lambda + (R_{21} - F_{21}R_{11}) & (B_{21} - F_{21}B_{11})\lambda + (L_{21} - F_{21}L_{11}) \end{Vmatrix} = n - m_1 \qquad (2.2)$$

$$\forall \lambda \in \Lambda, \quad \Lambda = \left\{ \lambda_i \colon \det \begin{Vmatrix} A_0\lambda^2 + D_0\lambda + W_0 & C_0\lambda - P_0^T \\ C_0^T\lambda^2 + S_0\lambda + R_0 & B_0\lambda + L_0 \end{Vmatrix} = 0 \right\}$$

$$\forall \lambda \in \Lambda^+, \quad \Lambda^+ = \{ \lambda \in \Lambda,\ \operatorname{Re}\lambda \geqslant 0 \}$$

is satisfied.

Here the matrices

$$C_0^T = \begin{Vmatrix} C_{11}^T \\ C_{21}^T \end{Vmatrix}, \quad S_0 = \begin{Vmatrix} S_{11} \\ S_{21} \end{Vmatrix}, \quad B_0 = \begin{Vmatrix} B_{11} \\ B_{21} \end{Vmatrix}, \quad R_0 = \begin{Vmatrix} R_{11} \\ R_{21} \end{Vmatrix}, \quad L_0 = \begin{Vmatrix} L_{11} \\ L_{21} \end{Vmatrix}$$

$$C_1^T(m_1 \times r), \quad C_{21}^T((m - m_1) \times r), \quad B_{11}(m_1 \times m), \quad B_{21}((n - m_1) \times m)$$

are divided into blocks in accordance with representation (2.1).

In order to prove the theorem it is necessary to rewrite system (1.4) in Cauchy form [taking into account relations (2.1)]

$$M_z z^{\cdot} = A_z z + B_z v_1 \qquad (2.3)$$

$$M_z = \begin{Vmatrix} E_r & 0 & 0 \\ 0 & A_0 & C_0 \\ 0 & C_0^T & B_0 \end{Vmatrix} \qquad A_z = \begin{Vmatrix} 0 & E_r & 0 \\ -W_0 & -D_0 & P_0^T \\ -R_0 & -S_0 & -L_0 \end{Vmatrix}$$

$$B_z = \| 0 \ \ 0 \ \ F_v^T \|^T, \quad z = \| x^T \ \ x^{\cdot T} \ \ \eta^T \|^T$$

and to use the following criterion of controllability [8]: system (2.3) is controllable if and only if

$$\operatorname{rank}(A_z - \lambda M_z,\ B_z) = n + r, \quad \forall \lambda \in \Lambda$$

$$\Lambda = \{ \lambda_i \colon \det(A_z - \lambda M_z) = 0 \}$$

If the set Λ has a zero root of multiplicity ν, i.e.

$$\operatorname{rank} \begin{Vmatrix} W_0 & -P_0^T \\ R_0 & L_0 \end{Vmatrix} = n - \nu$$

then the proof of Theorem 1 implies the following statement: for system (1.4) to be controllable, when its characteristic equation has zero ropot of multiplicity ν, it is necessary that the dimensions of the vector of controls should satisfy the condition $m_1 \geqslant \nu$.

Note that this statement for controlling a mechanical system with cyclic coordinates in a neighbourhood of a steady motion corresponds to the statement for controlling a mechanical system in a neighbourhood of an equilibrium position [9, 10]. When the number of controls is equal to the number of cyclic coordinates ($m_1 = m$, $F_v = F_m$) Theorem 1 implies the following theorem.

Theorem 2. System (1.4) of the $(n+r)$th order is controllable (stabilizable) if and only if the condition

$$\begin{gathered}\operatorname{rank}\| A_0\lambda^2 + D_0\lambda + W_0 \quad C_0\lambda - P_0^T \| = r \\ \forall\lambda\in\Lambda_1 \quad (\forall\lambda\in\Lambda_1^+) \\ \Lambda_1 = \{\lambda_i \quad \det(A_0\lambda^2 + D_0\lambda + W_0) = 0\}\end{gathered} \tag{2.4}$$

is satisfied.

Comparing condition (2.4) with the criteria of controllability and observability for systems of the second order [11], we will formulate a statement which enables us to reduce the investigation of the controllability of the original system (1.4) of the $(m+2r)$the order to an analysis of the observability of a certain system of the $2r$th order.

Theorem 3. System (1.4) is controllable if and only if the system

$$A_0 y^{\cdot\cdot} + D_0^T y^{\cdot} + W_0^T y = 0, \quad \sigma = -P_0 y + C_0^T y^{\cdot} \tag{2.5}$$

of the $2r$th order is observable.

Note that the criteria of another type can be used to analyse the observability of system (2.5). In particular, if the constraints imposed on the system are steady and $Q_k^N = 0$ ($k = r+1, \ldots, n$), $Q_j^N = Q_{jd}$ ($j = 1, \ldots, r$), the theorems formulated in [3] follow from Theorems 2 and 3.

For a GUS ($C_0 = 0$) Theorem 3 may be formulated as follows.

Corollary 3.1. GUS (1.4) of the $(2r+m)$th order is controllable is and only if the system

$$A_0 y^{\cdot\cdot} + D_0 y^{\cdot} + W_0 y = P_0^T u$$

of the $2r$th order is controllable.

For a gyroscopically coupled system (GCS), when $P_0 = 0$, from Theorems 2 and 3 one can deduce the following.

Corollary 3.2. If the matrix P_0 satisfies the condition $P_0 = 0$ for GCS (1.4), this system is controllable if and only if $\det W_0 \neq 0$ and the system

$$A_0 y^{\cdot\cdot} + D_0 y^{\cdot} + W_0 y = C_0 u$$

is controllable.

Corollary 3.3. If there is one positional coordinate ($r = 1$) in the system, then system (1.4) is controllable if and only if

$$\lambda C_0 \neq P_0^T \quad \forall\lambda\in\Lambda_1$$

3. Let us consider the question of the observability of system (1.4).

In the general case, when the measurements are in form (1.5) and $H_i \neq 0$ ($i = 1, 2, 3$) the conditions of observability may be formulated as follows.

Theorem 4. System (1.4) of the $(n+r)$th order is observable with respect to measurement (1.5) if and only if the conditions

$$\operatorname{rank} N = n \quad \forall \lambda \in \Lambda \tag{3.1}$$

$$N = \begin{Vmatrix} A_0\lambda^2 + D_0\lambda + W_0 & C_0\lambda - P_0^T \\ C_0\lambda^2 + S_0\lambda + R_0 & B_0\lambda + L_0 \\ H_2\lambda + H_1 & H_3 \end{Vmatrix}$$

are satisfied.

If among the roots of the characteristic equation of system (1.4) there is a zero root of multiplicity ν, i.e. condition (2.4) is satisfied, then Theorem 4 implies the following statements.

Corollary 4.1. In the case when Λ has a zero root of multiplicity ν, for system (1.4) to be observable it is necessary that the number l of measurements should satisfy the condition $l \geq \nu$.

Corollary 4.2. System (1.4) is not observable with respect to the measurement $\sigma = H_1 x + H_2 x^{\bullet}$ if $P_0 = 0$ and $L_0 = 0$.

Suppose that only the positional coordinates ($H_2 = 0$, $H_3 = 0$ and rank $H_1 = l < r$) or the positional velocities ($H_1 = 0$, $H_3 = 0$, rank $H_2 = l < r$) are measured. By modifying the measurements we can represent the matrix H_1 as $H_1 = \|E_l H_{12}\|$ in the first case and the matrix H_2 as $H_2 = \|E_l H_{22}\|$ in the second case.

Consider the matrix

$$N_i = \begin{Vmatrix} (A_{12} - A_{11}H_{i2})\lambda^2 + (D_{12} - D_{11}H_{i2})\lambda + \\ \quad + (W_{12} - W_{11}H_{i2}) & C_0\lambda - P_0^T \\ (C_{12}^T - C_{11}^T H_{i2})\lambda^2 + (S_{12} - S_{11}H_{i2})\lambda + \\ \quad + (R_{12} - R_{11}H_{i2}) & B_0\lambda + L_0 \end{Vmatrix}$$

Corollary 4.3. System (1.4) of the $(n + r)$th order is observable with respect to the measurement $\sigma = H_1 x$ of the positional coordinates if and only if the condition

$$\operatorname{rank} N_1 = n - l \ \ \forall \lambda \in \Lambda \text{ holds.}$$

Corollary 4.4. System (1.4) of the $(n + r)$th order is observable with respect to the measurement $\sigma = H_2 x^{\bullet}$ of the positional velocities if and only if the condition

$$\operatorname{rank} N_2 = n - l \quad \forall \lambda \in \Lambda,$$

holds, and the set Λ has no values $\lambda = 0$.

Unlike the case of time-independent constraints and the absence of non-potential forces acting in the cyclic coordinates, when the set Λ always has zero roots and the system is not observable with respect to the measurement of $\sigma = H_2 x^{\bullet}$ [3], here, as Corollary 4.4 shows, non-observability is possible.

Consider the case when only the cyclic velocities are measured

$$H_1 = H_2 = 0, \qquad \operatorname{rank} H_3 = l < m$$

If we represent the matrix H_3 in the form $H_3 = \|H_{31} E_l\|$ and make the corresponding decomposition of the matrices B_0, L_0 and P_0, C_0 into the blocks of the form

$$\| \Phi_{21} \ \ \Phi_{22} \| \text{ and } \| \psi_{21} \ \ \psi_{22} \|$$
$$(\Phi_{21}(m \times (m - l)), \quad \Phi_{22}(m \times l), \quad \psi_{21}(r \times (m - l)), \quad \psi_{22}(r \times l))$$

we can formulate the following corollary.

Corollary 4.5. System (1.4) of the $(n + r)$th order is observable with respect to the measurement $\sigma = H_3 \eta$ of the cyclic coordinates if and only if

$$\operatorname{rank}\left\|\begin{matrix} A_0\lambda^2+D_0\lambda+W_0 & (-P_{21}^T+P_{22}^T H_{31})+(C_{21}-C_{22}H_{31})\lambda \\ S_2\lambda+R_2 & (B_{21}-B_{22}H_{31})\lambda+(L_{21}-L_{22}H_{31}) \end{matrix}\right\| = n-l \quad \forall\lambda\in\Lambda$$

Here $S_2 = S_0 - C_0^T A_0^{-1} D_0$ and $R_2 = R_0 - C_0^T A_0^{-1} W_0$.
This statement may be proved using the equality

$$\left\|\begin{matrix} A_0\lambda^2+D_0\lambda+W_0 & C_0\lambda - P_0^T \\ C_0^T\lambda^2+S_0\lambda+R_0 & B_0\lambda+L_0 \\ 0 & H_3 \end{matrix}\right\| =$$

$$= \left\|\begin{matrix} E_r & 0 & 0 \\ C_0^T A_0^{-1} & E_m & 0 \\ 0 & 0 & E_l \end{matrix}\right\| \left\|\begin{matrix} A_0\lambda^2+D_0\lambda+W & -P_0^T+C_0\lambda \\ S_2\lambda+R_2 & B_0\lambda+L_0 \\ 0 & H_3 \end{matrix}\right\|$$

Let us consider the cases, each taken separately, in which either all the positional coordinates ($\sigma = x$) or all the positional velocities ($\sigma = x^{\bullet}$), or all the cyclic velocities ($\sigma = \eta$) are measured. The advisability of such a consideration is due to the fact that in all these cases the investigation of observability of system (1.4) of the $(n+r)$th order reduces to investigating the observability of a system of lower order.

The following theorems hold.

Theorem 5. System (1.4) of the $(2r+m)$th order is observable with respect to the measurement of all the positional coordinates if and only if the system

$$B_0\chi^{\bullet} + L_0\chi = 0, \quad \sigma = P_2^T\chi \qquad (3.2)$$
$$P_2^T = P_0^T + C_0 B_0^{-1} L_0$$

of the mth order is observable.

System (3.2) is observable if and only if the condition [8]

$$\operatorname{rank}\left\|\begin{matrix} P_2^T \\ B_0\lambda+L_0 \end{matrix}\right\| = m, \quad \forall\lambda\in\Lambda_2$$
$$\Lambda_2 = \{\lambda:\ \det(B_0\lambda+L_0)=0\}$$

is satisfied.

Theorem 6. System (1.4) is observable with respect to the measurement of all the positional velocities if and only if the condition

$$\det\left\|\begin{matrix} W_0 & -P_0^T \\ R_0 & L_0 \end{matrix}\right\| \neq 0$$

is satisfied and system (3.2) of the mth order is observable.

Theorem 7. System (1.4) of the $(2r+m)$th order is observable with respect to the measurement of all the cyclic coordinates if and only if the system $A_0\chi^{\bullet\bullet} + D_0\chi^{\bullet} + W_0\chi = 0$, $\sigma = S_2\chi^{\bullet} + R_2\chi$ of the 2rth order is observable.

This statement is equivalent to the condition [11]

$$\operatorname{rank}\left\|\begin{matrix} A_0\lambda^2+D_0\lambda+W_0 \\ S_2\lambda+R_2 \end{matrix}\right\| = r, \quad \forall\lambda\in\Lambda_1$$

These theorems may be proved by representing system (1.4) in the form (2.3) and using the theorem [12] on the equivalence of the observability conditions for system (2.3) with the measurement $\sigma = H_2 z$ and for the system

$$M_z z = (A_z - K_z H_z) z + B_z v_1, \qquad \sigma = H_z z$$
$$(K_z = \text{const})$$

When the conditions of controllability and observability of systems (1.4) and (1.5), which are derived from the theorems formulated above, have been stated, it is possible to set up algorithms of stabilization by analogy with [3].

4. *Example.* Let us consider the problem of stabilizing the steady motions of a symmetric rotor of mass m which is fixed at the middle of an elastic shaft with a certain eccentricity l. Here, as in many publications (see, for instance, [13–15]), we assume that the motions of the rotor are plane. We take the point O of the intersection of the plane of the motion and the straight line drawn between the centres of supports of the shaft as the origin of the moving system of coordinates. The x_1 axis is parallel to the segment $G\epsilon(|G\epsilon| = l)$ connecting the centre of mass G of the rotor and the point ϵ of the rotor's fixing on the shaft. Let x_1 and x_2 denote the coordinates of the centre of mass G, and let φ be the angle between the x_1 axis and a fixed direction ξ_1.

The kinetic energy of the rotor and the force function of the elastic coupling have the form [14]

$$T = \tfrac{1}{2}m[x_1^{\cdot 2} + x_2^{\cdot 2} + 2\varphi^{\cdot}(x_1 x_2^{\cdot} - x_2 x_1^{\cdot}) +$$
$$+ \varphi^{\cdot 2}(x_1^2 + x_2^2)] + \tfrac{1}{2}J\varphi^{\cdot 2}$$
$$U = \tfrac{1}{2}[c_1 x_2^2 + c_2(x_1 + l)^2]$$

Here c_1 and c_2 are the principal stiffnesses of the shaft. We denote the central moment of inertia of the rotor by $J = m\rho^2$ where ρ is the radius of inertia of the rotor.

Let us assume that the forces of internal and external friction act on the rotor, supposing, as this takes place, that the force of external friction is proportional to the absolute speed of the centre of mass and the force of internal friction is proportional to the relative speed [14].

The system under consideration is gyroscopically coupled; x_1 and x_2 are the positional coordinates and φ is the cyclic one. The generalized controlling force (the moment of drive) corresponding to the coordinate φ must be specified.

The equations of motion have the following special solution describing the steady motion of the rotor

$$x_{10} = c_1\kappa_2 l/\Delta, \quad x_{20} = -c_1 a\omega_0 l/\Delta, \quad \varphi^{\cdot} = \omega_0 = \text{const}$$
$$\Delta = \kappa_1\kappa_2 + a^2\omega_0^2, \quad \kappa_j = c_j - \omega_0^2 \qquad (j = 1, 2) \tag{4.1}$$

The moment F_0 corresponding to steady motion (4.1) is specified by the relation $F_0 = a\omega_0(x_{10}^2 + x_{20}^2)$ In particular, when $a = 0$, steady motion (4.1) exists if $k_j \neq 0$.

It is well known [14] that a range of values of the angular velocity ω_0 exists in which the steady motion (4.1) is unstable, and it is necessary to introduce an additional control in order to stabilize it.

The linearized equations of perturbed motion of the rotor have the form (1.4) in which $x = \|x_1 x_2\|^T$, η is a scalar, and

$$A_0 = E_2, \quad D_0 = (a + b)E_2 + 2\omega_0 I_2, \quad W_0 = \begin{Vmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{Vmatrix} + a\omega_0 I_2$$

$$C_0^T = \| -x_{20} \quad x_{10} \|, \quad P_0 = S_0 = \| 2\omega_0 x_{10} - a x_{20}, \quad 2\omega_0 x_{20} + a x_{10} \|$$

$$B_0 = \rho^2, \quad R_0 = \| 2a\omega_0 x_{10} \quad 2a\omega_0 x_{20} \|, \quad L_0 = a r_0^2$$

$$r_0^2 = x_{10}^2 + x_{20}^2, \quad F_0 = 1, \quad I_2 = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}$$

The controllability condition given by Theorem 2 for this problem has the form

$$[\lambda^2 + (a + b)\lambda + \kappa_1][\lambda^2 + (a + b)\lambda + \kappa_2] + \omega_0^2(2\lambda + a)^2 \neq 0$$
$$\lambda = -\frac{\omega_0 r_0^2(2\kappa_1 + a^2) + \epsilon x_{10}(2\omega_0 x_{10} - a x_{20})}{\omega_0 r_0^2(3a + 2b) + \epsilon x_{10} x_{20}}, \quad \epsilon = \kappa_2 - \kappa_1 \tag{4.2}$$

Condition (4.2) is broken down only when a certain relation exists between the parameters of the system. We note a number of special cases.

1. A circular shaft ($k_1 = k_2 = k$, $\epsilon = 0$). The condition for controllability to break down is

$$\kappa = \tfrac{1}{2}a(\tfrac{1}{2}a + b) \tag{4.3}$$

In particular, when $a = 0$, the system is always controllable in a neighbourhood of the motion (4.1). Note that in [16] where only the case $a = 0$, $k_1 = k_2$ was considered, due to an inaccuracy, a fictious condition for controllability to break down occurred.

2. There is no external friction ($a = 0$). The condition for controllability to break down has the form

$$\kappa_1 = \kappa_2 - 4\omega_0^2 - \kappa_2^2/b^2 \tag{4.4}$$

Condition (4.4) can only be satisfied when $k_1 \neq k_2$ and $b > \sqrt{c_1 + 3\omega_0^2}$.

It can be shown that system (1.4) for the problem under consideration is always observable when the positional coordinates (x_1, x_2) are measured. If the positional velocities $(\dot{x}_1, \dot{x}_2)$ are measured this always holds except when the parameters of the system satisfy the condition

$$a[(\kappa_1\kappa_2 + 3a^2\omega_0^2)r_0 + 4\omega_0^2(\kappa_2 x_{10}^2 + \kappa_1 x_{20}^2) - 2a\omega_0 x_{10} x_{20}\epsilon] = 0$$

Obviously, if there is no friction ($a = 0$), the system is non-observable.

According to Theorem 7 the system under consideration is always observable with respect to the cyclic velocity ($\sigma = \eta$) except for the following cases

1. $(\kappa_1 - ab)(\kappa_2 - ab) + a^2\omega_0^2 = 0$
2. $[\lambda^2 + (a+b)\lambda + \kappa_1][\lambda^2 + (a+b)\lambda + \kappa_2] + \omega_0^2(2\lambda + a)^2 = 0$

$$\left(\lambda = -\frac{2\omega_0(r_0^2\kappa_1 + \epsilon x_{10}^2)}{(a+2b)\omega_0 r_0^2 - \epsilon x_{10} x_{20}}\right).$$

For a circular shaft the condition for observability to break down agrees with condition (4.3). It agrees with condition (4.4) if there is no external friction.

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